Monomials (terms) and polynomials

A **monomial** is a collection of a numbers and some letters all multiplied together. The outcoming number is called the **numerical coefficient**, the letters are the **variable part.** The **degree of a monomial** is the number of letters of the variable part.

Numbers are thought as zero degree monomials, also called **constant terms.**

Examples:

$3xyz$ .... degree 3 .... coefficient 3
$4z^2x$ .... degree 3 .... coefficient 4
$\frac{xy}{2}$ .... degree 2 .... coefficient 1/2
8 .... degree 0 .... coefficient 8 .... it is a constant term
$\frac{5}{3}x$ .... degree 1 .... coefficient 5/3
$-7x^2$ .... degree 2 .... coefficient – 7
$-6x$ .... degree 1 .... coefficient – 6
$\left(\frac{1}{2}x\right)^2$ .... degree 2 .... coefficient 1/4
$7x^3$ .... degree 3 .... coefficient 7
$\frac{5}{3}x^2$ .... degree 2 .... coefficient 5/3
$\frac{2}{3}x\cdot 4x^2$ .... degree 3 .... coefficient 8/3

Like monomials are those with the same variable part.

Example: $-7x^2$, $\left(\frac{1}{2}x\right)^2$ and $\frac{5}{3}x^2$ are like monomials

A **polynomial** is a variable expression that consist of a sum of one or several monomials. The **degree of a polynomial** is the greater of the degrees of its monomials. The monomials of a polynomial are also called **terms.**

Example: $4x^2 + 19xy + (-3y^3) + 12$ has four terms: $4x^2$, $19xy$, $-3y^3$ and 12
their degrees are 2, 2, 3 and 0 respectively
so the polynomial's degree is 3

A **simplified polynomial** does not contain any like terms. In such a polynomial we can easily spot the degree and a single constant term. The numerical coefficient of the biggest degree term is called in spanish **"coeficiente principal"**

Note: the **terms** in a variable expression are those parts that are separated by addition symbols (+ or –) that is, the addends; the **variable terms** have got letters but the **constant terms** are just numbers without variable part; **like terms** are those terms that have the same variable part.

Note: a polynomial having two terms is called a **binomial**

Note: combining like terms by adding their numerical coefficients is called **simplifying a variable expression.**

Note: the solutions of $a_nx^n + ... + a_2x^2 + a_1x + a_0 = 0$ are called the **roots of** $a_nx^n + ... + a_2x^2 + a_1x + a_0$

Multipying monomials

If a monomial is to be multiplied by a numerical multiplier, the coefficient alone is multiplied.

Examples: $5 \cdot 3abc = 15abc$

$x \cdot 2a = 2ax$ (note that the numerical factor is interchanged with literal factors)

Rule for multiplication of monomials: multiply the numerical coefficients and multiply the literal factors.

Example: $2ab \cdot 3a^2 \cdot 2b^3 = 2 \cdot 3 \cdot 2 \cdot a \cdot a^2 \cdot b^3 = 12a^3b^4$

Dividing monomials

When the divisor is numerical, divide the coefficient of the dividend by the divisor.

Example: $6a : 3 = 2a$

When the divisor contains literal parts that are also in the dividend, cancellation may be used.

Examples: $6ab : (2a) = \frac{6ab}{2a} = \frac{3b}{1} = 3b$

$6a^2b : (15b^3) = \frac{6a^2b}{15b^3} = \frac{2a^2}{5b^2}$ or else $6a^2b : (15b^3) = \frac{6}{15}a^2b^{1-3} = \frac{2}{5}a^2b^{-2}$

Powers and roots of monomials

Apply power rules and surd rules.

Examples: $(2x^3)^5 = 2^5 \cdot (x^3)^5 = 32x^{15}$

$\sqrt[3]{125x^9} = \sqrt[3]{125} \cdot \sqrt[3]{x^9} = 5x^3$

Adding and substracting monomials

Only like terms can be added or substracted. To do so, distributive property must be applied.

Examples: $12a + 4a - 5a = (12 + 4 - 5)a = 11a$ the three terms were added/substracted

$-5x^3 + 12x^3 + x = 7x^3 + x$ a term could not be added

Combining like terms by adding their numerical coefficients is called **simplifying a variable expression.**

Examples: Simplify $3x^2 + 5x + 2xy + 2x^2 - x$ We add like terms and get $5x^2 + 4x + 2xy$

Simplify $12a + 3 - 5a + 2$ We add like terms and get $7a + 5$

Multiplying a polynomial by a monomial

The distributive property is used. Multiply each term of the polynomial by the monomial.

Examples: $9(3 + z) = (9 \cdot 3) + (9 \cdot z) = 27 + 9z$

$8(c - 5) = (8 \cdot c) + [8 \cdot (-5)] = 8c + (-40) = 8c - 40$

$4a(2 + b) = (4a \cdot 2) + (4a \cdot b) = 8a + 4ab$

$14 - 5(2s + 2) + 3s = 14 + (-5) \cdot 2s + (-5) \cdot 2 + 3s = 14 - 10s - 10 + 3s = -7s + 4$

Removing a common factor

The distributive property is used again but in the opposite way.

Take account of which variables are repeated in all the terms; look at the factors of the numerical coefficients and do the same. Write them apart (they are the common factor). Leave inside brackets the quotients of dividing each monomial by the common factor.

If the product were calculated, the original expression should be recovered.

Examples: $x^2y - xy^2 = xy(x - y)$ and conversely $xy(x - y) = xy \cdot x - xy \cdot y = x^2y - xy^2$

$3ax^2 - 3ax = 3ax(x - 1)$ and conversely $3ax(x - 1) = 3ax \cdot x - 3ax \cdot 1 = 3ax^2 - 3ax$

Adding and substracting polynomials

Addition is done by grouping like monomials. Substraction is done the same way, but changing signs on the substraend previously.

Examples: $(x^4-3x^2+5x-1)+(2x^2-6x+3)+(2x^4+x^3-x-4)=$

$$\begin{array}{r} x^4+0x^3-3x^2+5x-1 \\ 0x^4+0x^3+2x^2-6x+3 \\ 2x^4+x^3+0x^2-x-4 \\ \hline 3x^4+x^3-x^2-2x-2 \end{array}$$

$(x^4-3x^2+5x-1)-(2x^2-6x+3)=$

$$\begin{array}{r} x^4+0x^3-3x^2+5x-1 \\ 0x^4+0x^3-2x^2+6x-3 \\ \hline x^4+0x^3-5x^2+11x-4 \end{array}$$

Multiplying polynomials

The distributive property is used. Multiply each term of one polynomial by each term of the other; simplify afterwards.

Examples: $(x^2-5x-1)\cdot(x-2)=$

$$\begin{array}{r} x^2-5x-1 \\ x-2 \\ \hline -2x^2+10x+2 \\ x^3-5x^2-x \\ \hline x^3-7x^2+9x+2 \end{array}$$

$(3x^3-5x^2+6)\cdot(2x+1)=$

$$\begin{array}{r} 3x^3-5x^2+6 \\ 2x+1 \\ \hline 3x^3-5x^2+6 \\ 6x^4-10x^3+12x \\ \hline 6x^4+7x^3-5x^2+12x+6 \end{array}$$

Quadratics

We multiply binomials like this: $(A+B)(C+D)=AC+AD+BC+BD$ (multiply out brackets)

We apply the following rules to multiply binomials in special cases:

$(A+B)^2=A^2+B^2+2AB$ (squared brackets)

$(A-B)^2=A^2+B^2-2AB$ (squared brackets)

$(A+B)(A-B)=A^2-B^2$ (difference of two squares)

You can easily proof they are right by multiplying out brackets and cancelling.

Examples: $(x-11)^2=x^2+11^2-2\cdot x\cdot 11=x^2+121-22x$

$(2x+1)^2=(2x)^2+1^2+2\cdot(2x)\cdot 1=4x^2+1+4x$

$\left(\frac{2}{5}x-5\right)^2=\left(\frac{2}{5}x\right)^2+5^2-2\cdot\left(\frac{2}{5}x\right)\cdot 5=\frac{4}{25}x^2+25-4x$

$(x^2+1)(x^2-1)=(x^2)^2-1^2=x^4-1$

$\left(\frac{x}{2}+y\right)\left(\frac{x}{2}-y\right)=\left(\frac{x}{2}\right)^2-y^2=\frac{x^2}{4}-y^2$

Polynomial division (long division)

We can proceed exactly like with natural numbers division. To get each new term of the quotient we must divide the highest degree monomial of the dividend by the highest degree monomial of the divisor.

Examples: $(4x^5+6x^4+2x^3+5x^2+3x+6):(2x^2+x)$

$$\begin{array}{l|l} 4x^5+6x^4+2x^3+5x^2+3x+6 & 2x^2+x \\ -(4x^5+2x^4) & 2x^3+2x^2+2{,}5 \quad \text{is the quotient} \\ \quad 4x^4+2x^3+5x^2+3x+6 & \\ \quad -(4x^4+2x^3) & \\ \qquad 5x^2+3x+6 & \\ \qquad -(5x^2+2{,}5x) & \\ \qquad 0{,}5x+6 \quad \text{is the remainder} & \end{array}$$

$(x^4+2x^3-x^2+x-4):(x-7)$

$$\begin{array}{l|l} x^4+2x^3-x^2+x-4 & x-7 \\ -(x^4-7x^3) & x^3+9x^2+62x+435 \quad \text{is the quotient} \\ \quad 9x^3-x^2+x-4 & \\ \quad -(9x^3-63x^2) & \\ \qquad 62x^2+x-4 & \\ \qquad -(62x^2-434x) & \\ \qquad\quad 435x-4 & \\ \qquad\quad -(435x-3045) & \\ \qquad\quad 3041 \quad \text{is the remainder} & \end{array}$$

Ruffini's rule

It is only suitable when divisor is the type (x-a). You have to build a box and write the coefficients of the dividend in decreasing order of degree, without forgetting the zero coefficients. The number "a" is written out of the box, to the left. The first coefficient is placed below the box and the algorithm starts (multiply by "a", place the product on the next column, add that column, place the sum below the box and repeat all the steps again). The numbers below the box are the quotient coefficients, and the last of them is the remainder:

Examples: $(x^4+2x^3-x^2+x-4):(x-7)$

	1	2	−1	1	−4
7		7	63	434	3045
	1	9	62	435	3041

3041 is the remainder

$x^3+9x^2+62x+435$ is the quotient

$(x^4-5x^3-49x^2+2x-3):(x+5)$

	1	−5	−49	2	−3
−5		−5	50	−5	15
	1	−10	1	−3	12

12 is the remainder

x^3-10x^2+x-3 is the quotient

EXERCISES

1) (a) If $x = 3$, find the value of
$5x + 1$
(b) If $y = 7$, evaluate $\sqrt{5y + 1}$
(c) Simplify
$3(2a + 5b) + 2(3a + 7b)$

2) (a) If $a = 4$ and $b = -5$ find the value of:
(i) $a + b$
(ii) $a - b$
(b) Simplify $3x - 4y + 11x - y$
(c) Multiply out $(a + 7)(a + 6)$

3) (a) If $p = 5$ and $q = -7$, evaluate
$\frac{p - q}{p + q}$
(b) Multiply out $(2x - 11)(x + 7)$
(c) If $a = 3$, $b = 4$ and $c = -5$, evaluate:
(i) $a + b + c$
(ii) $a - b - c$
(iii) $\frac{a^2 + b^2}{c^2}$
(iv) $\sqrt{2ab - 8c}$
(v) $\frac{2abc}{2(a + b - c)}$

4) (a) If $x = 3$, find the value of
$x^2 - 8x + 15$
(b) Simplify
$3(x + 3y - z) - (x - y - 5z)$
(c) Multiply out
$(2x - 5)(x^2 - x - 2)$

5) (a) If $y = 5$, find the value of
$2y^2 - y + 10$
(b) Multiply out $(3x^2y)(8xy^2)$
(c) Simplify $(2x - 1)^2 - (x + 2)^2$

6) (a) Multiply out
$(2x + 3)(x^2 - x + 10)$
(b) If $x = 9$, evaluate
$x^2 - 10\sqrt{x} + 5$

7) **Simplify the following:**

1. $3(2a+5b)$
2. $4(6b-c)$
3. $7(3x-2y)$
4. $-2(2x+7y)$
5. $-3(x-y)$
6. $-10(2x-5y)$
7. $-(x-y+z)$
8. $x(2x+5)$
9. $a(12a-4)$
10. $3a(7a-2b)$
11. $x(x^2+x+1)$
12. $2(x+3y)+3(2x-y)$
13. $2(5x^2-x-2)+3(x^2+x-4)$
14. $2x(x-9)-x(x+7)$
15. $5(x-2y)-3(x+y)$
16. $a^2(a^3+a^2+a+1)$
17. $3(x+4y+1)-(x+y-11)$
18. $x(5x-1)-(5x^2-x-10)$
19. $x^2(7x-2)-3x(x^2-x-1)$
20. $\frac{1}{2}(10x-8y+6)-(5x-4y+3)$

8) **If $p = x^2-x+2$ and $q = x^2+3x-3$, write the following in terms of x:**

(i) $p+q$ (ii) $p-q$
(iii) $3p+q$ (iv) $3p-2q$

9) **If $a = x+3y-5$ and $b = 2x-y+2$, write the following in terms of x and y:**

(i) $a+b$ (ii) $a-b$
(iii) $2a+5b$ (iv) $2a-b$

10) Copy this test and complete the 7 unfinished words:

When a letter, say x, is used to represent an unknown number, we call the letter a v . A number whose value does not change is called a c . $11x^2$, $2ab$, x^3 are all examples of t s. When some of these are combined (e.g. $3x^2+4xy-y^2$), we have an e
The number 11 in $3x^3+11x^2-4x+7$ is called the c of x^2.
Terms which have the same variables to the same power ($3x^2y$ and $8x^2y$, for example) are called l terms. We can add and subtract such terms. But we cannot add or subtract terms which are u .

11) Multiply out and simplify:

$(2x-1)(3x+4)$	$(a+b)(x+y)$
$(x-3)(x+5)$	$(2x-y)(p+4q)$
$(3x+2)(2x-5)$	$(a-b)(b-d)$
$(3x-1)(4x-1)$	$(2x+1)(4x^2-2x+1)$
$(5p-1)(4p-3)$	$(x+1)^3$
$(2a-b)(3a-4b)$	$(x-2)^3$
$(2b+7)^2$	$(x+1)(x+2)+(x+4)(x+5)$
$(x+12)^2$	$(a+3)(a-4)+(2a+1)(a-5)$
$(5t-4)^2$	$3(y+1)^2-4(2y-1)$
$2(x-6)^2$	$(p+3)(p-2)-(p-6)(p-1)$
$(2x+1)(x^2+x+1)$	$(x+y)^2-2xy$
$(x+3)(2x^2+3x+5)$	$(a+b)(x-y)-(a-b)(x+y)$
$(3a+2)(5a^2-a-5)$	$(x+1)(x+2)(x-1)$
$(x-1)(x^2+x+1)$	$(2x-y)(2x+y)(x+y)$
$(3y^2+y-7)(y+2)$	$(x-3)^2-(x+3)^2$
$(2a-3b)(5a^2-2ab+b^2)$	$(2x-1)^2-4(x+1)^2$

12) Multiply out and simplify:

a) $x(x^2+1)-3x(-x+3)+2(x^2-x)^2$
b) $2(x^2+3)-2x(x-3)+6(x^2-x-1)$
c) $-4x(x-4)^2+3(x^2-2x+3)-2x(-x^2+5)$
d) $-3x(x+7)^2+(2x-1)(-3x+2)$
e) $(2a^2+a-1)(a-3)-(2a-1)(2a+1)$
f) $(3b-1)(3b+1)-(4b-3)^2-2(2b^2+16b-16)$

13) Divide:

a) $(12x^3-3x):(3x)=$
b) $(2x^4+12x^3+18x^2):(2x^2)=$
c) $(45x^2-120x+80):(5)=$
d) $(3x^3-15x):(3x)=$

14) Divide:

a) $(x^5+7x^3-5x+1):(x^3+2x)$
b) $(x^3-5x^2+x):(x^2-1)$
c) $(x^3-5x^2+x):(2x^2-1)$

15) Divide using "Ruffini's rule":

a) $(2x^3 - x^2 + 5x - 3) : (x - 2)$
b) $(-x^4 + 3x^2 - 2x + 1) : (x + 1)$
c) $(3x^3 + 5x^2 - x) : (x + 2)$
d) $(x^3 - 27) : (x - 3)$
e) $(x^4 - x^2) : (x + 1)$
f) $(x^5 - 2x^4 + x - 2) : (x - 1)$

16) Divide:

(i) $x^2 + 12x + 20$ by $x + 2$
(ii) $x^2 - 2x - 63$ by $x - 9$
(iii) $12a^2 + 14a + 4$ by $3a + 2$
(iv) $15c^2 + 11c - 14$ by $3c - 2$
(v) $2x^2 - x - 21$ by $2x - 7$
(vi) $77x^2 + 10x - 3$ by $11x + 3$
(vii) $2x^2 - xy - 6y^2$ by $2x + 3y$
(viii) $6s^2 - 11st - 35t^2$ by $2s - 7t$
(ix) $2 + x - 21x^2$ by $1 - 3x$
(x) $4 - 12x - 55x^2$ by $2 + 5x$

17) Perform these divisions:

(i) $(x^3 + 3x^2 + 5x + 3) \div (x + 1)$
(ii) $(x^3 + 5x^2 + 11x + 10) \div (x + 2)$
(iii) $(x^3 - 3x^2 + 3x - 1) \div (x - 1)$
(iv) $(4b^3 - 4b^2 - 9b + 9) \div (2b - 3)$
(v) $(x^3 + x^2 - 7x + 5) \div (x - 1)$

18) Divide $x^3 - 19x - 30$ by $x - 5$.
(Hint: Write it as
$x^3 + 0x^2 - 19x - 30 \div x - 5$.)

19) Divide $x^3 + 3x^2 + 50$ by $x + 5$.
(Hint: Write this as
$x^3 + 3x^2 + 0x + 50 \div x + 5$.)

20) Divide $\dfrac{x^3 + 5x^2 + 324}{x + 9}$

21) Simplify $\dfrac{x^3 - 13x - 12}{x + 1}$ and factorise the simplified expression.

22) Divide $\dfrac{2k^3 + 9k^2 - 55k + 50}{2k - 5}$ and check your answer by letting $k = 3$

23) Simplify $\dfrac{2a^2(2a + 3) - 27}{2a - 3}$

24) Simplify $\dfrac{28 - 39a + 2a^3}{4 - a}$ and check your answer by letting $a = 2$.

Factorising polynomials

The **factors** of an expression are the parts which **multiply** together to give that expression.

Factorising is the opposite of multiplying out brackets. That is to write a polynomial as a product of two or several lower degree polynomials.

Computing factors of polynomials can be done:

- by removing common factors
- by means of different formulae (like "difference of two squares" or "squared brackets"
- by enchained divisions using Ruffini's rule (where the remainder is required to be zero)
- by solving equations (to be explained further)

- Here are two kinds of factor

  Common factor: $6a^2b - 8ab^2 = 2ab(3a-4b)$

  Grouped common factor:
  $$\begin{aligned} & 3a^2 - ab + 9ac - 3bc \\ &= a(3a-b) + 3c(3a-b) \\ &= (a+3c)(3a-b) \end{aligned}$$

  If the signs in a group-of-four are $+--+$ or $++--$, then we take a **negative** factor out of the second pair.

  $$\begin{aligned} ab - ac - bx + cx &= a(b-c) - x(b-c) \\ &= (a-x)(b-c) \end{aligned}$$

  $$\begin{aligned} a^2 + 3a - ab - 3b &= a(a+3) - b(a+3) \\ &= (a-b)(a+3) \end{aligned}$$

- $x^2 + y^2$ has no factors but $x^2 - y^2$ (which is called the **'difference of two squares'**) as factors:

  $x^2 - y^2 = (x-y)(x+y)$

- The method of dividing by binomials like (x – a) is based on the same reasoning we use to factorise a number in prime factors: $480=2^5\cdot 3\cdot 5$

480	2
240	2
120	2
60	2
30	2
15	3
5	5

- See page 6 ("Factorising 2nd degree polynomials")

Example

Factorise:

(i) $x^2 - 16x$

(ii) $x^2 - 16$

(iii) $x^3 - 16x^2 + 64x$

Solution

(i) $x^2 - 16x$ contains a 'common factor' x

$\therefore \; x^2 - 16x = x(x-16)$

(ii) $x^2 - 16$ is the 'difference of two squares'

$\therefore \; x^2 - 16 = (x-4)(x+4)$

(iii) $x^3 - 16x^2 + 64x$ has a 'common factor' x

$\therefore \; x^3 - 16x^2 + 64x = x(x^2 - 16x + 64)$

$= x(x-8)(x-8)$

Example

Factorise $x^2 + 2xy + y^2 - 49$

Solution

$x^2 + 2xy + y^2 - 49$

This is a quadratic.	This is on its own.

$$\begin{aligned} & x^2 + 2xy + y^2 - 49 \\ =\ & (x+y)(x+y) - 49 \\ =\ & (x+y)^2 - 49 \\ =\ & (x+y)^2 - (7)^2 \\ =\ & (x+y-7)(x+y+7) \end{aligned}$$

(Ah! This is the **difference of two squares**!)

Examples:

$x^2 + 3x = x(x+3)$ (removing a common factor)

$x^2 - 25 = x^2 - 5^2 = (x+5)(x-5)$ ("difference of two squares")

$4x^2 + 12x + 9 = (2x)^2 + 2(2x)3 + 3^2 = (2x+3)^2$ (applying "squared brackets")

$x^2 - 8x + 16 = x^2 - 2x4 + 4^2 = (x-4)^2$ (applying "squared brackets")

$x^3 - 6x^2 + 9x = x(x^2 - 6x + 9) = x(x^2 - 2x3 + 3^2) = x(x-3)^2$ (both methods)

$4a^2 - 81b^2 = (2a)^2 - (9b)^2 = (2a-9b)(2a+9b)$ ("difference of two squares")

$9x^2 - 25 = (3x)^2 - (5)^2 = (3x-5)(3x+5)$ ("difference of two squares")

Note:

Sometimes you will have to re-order the group:

$$\begin{aligned} & 10ps + q + 2sq + 5p \\ =\ & 10ps + 5p + 2sq + q \\ =\ & 5p(2s+1) + q(2s+1) \\ =\ & (5p+q)(2s+1) \end{aligned}$$

When you re-order the group, the **ratio** of the coefficients in each twosome should be the same. In the above example, note that $10:5 = 2:1$ and this is why we chose the new order $\mathbf{10}ps + \mathbf{5}p + \mathbf{2}sq + \mathbf{1}q$.

(The order $10ps + 2sq + 5p + 1q$ would also be satisfactory since $10:2 = 5:1$.)

Remember that 1 (or -1) is always available as a common factor.

$$\begin{aligned} & 3a^2 - 6ab - a + 2b \\ =\ & 3a(a-2b) - 1(a-2b) \\ =\ & (3a-1)(a-2b) \end{aligned}$$

Note:

Don't forget the CF rule: "common factors come first"

For example,

$2x^2 - 50 = 2(x^2 - 25) = 2(x-5)(x+5)$

$x^3 - 144x = x(x^2 - 144) = x(x-12)(x+12)$

Factorise $4x^3 - 6x^2 - 16x - 6$

EXAMPLE

dividing with Ruffini's rule

	4	−6	−16	−6
−1		−4	10	6
	4	−10	−6	0

divisor (x + 1)

quotient $4x^2 - 10x - 6$

using now quadratic formula

$$\frac{10 \pm \sqrt{100+96}}{8} = \begin{cases} 3 \\ -1/2 \end{cases}$$

so we can write

$$4x^3 - 6x^2 - 16x - 6 =$$
$$= (x+1)(4x^2 - 10x - 6) =$$
$$= (x+1)4(x-3)\left(x + \frac{1}{2}\right)$$

Example

Factorise $(5a + 6b)^2 - (a - b)^2$

Solution

$$(5a + 6b)^2 - (a - b)^2$$
$$= [(5a + 6b) - (a - b)][(5a + 6b) + (a - b$$
$$= [5a + 6b - a + b][5a + 6b + a - b]$$
$$= [4a + 7b][6a + 5b]$$

EXAMPLE

$x^4 - 8x^3 + 11x^2 + 32x - 60$ (dividing with Ruffini's rule)

		1	−8	11	32	−60
divisor (x − 2)	2		2	−12	−2	60
		1	−6	−1	30	0
divisor (x + 2)	−2		−2	16	−30	
		1	−8	15	0	
divisor (x + 2)	3		3	−15		
		1	−5	0		
divisor (x − 5)	5		5			
		1	0			

$x^3 - 6x^2 - x + 30$ is the quotient

$x^2 - 8x + 15$ is the quotient

$x - 5$ is the quotient

1 is the quotient

so $x^4 - 8x^3 + 11x^2 + 32x - 60 = (x-2)(x+2)(x+2)(x-5)$

Note: When you reach a quadratic which you know how to factorise by other means, you can stop dividing and factorise the other way.

EXAMPLE

$x^4 + 2x^3 - 7x^2 - 8x + 12$ (dividing with Ruffini's rule)

		1	2	−7	−8	12
divisor (x − 1)	1		1	3	−4	−12
		1	3	−4	−12	0
divisor (x − 2)	2		2	10	12	
		1	5	6	0	
divisor (x + 2)	−2		−2	−6	12	
		1	3	0		
divisor (x + 3)	−3		−3			
		1	0			

$x^3 + 3x^2 - 4x - 12$ is the quotient

$x^2 + 5x + 6$ is the quotient

$x + 3$ is the quotient

1 is the quotient

so $x^4 + 2x^3 - 7x^2 - 8x + 12 = (x-1)(x-2)(x+2)(x+3)$

Note: When you reach a quadratic without roots, it remains as the last factor of the factorised expression.

EXERCISES

25) **Copy these and fill the brackets, but watch the signs carefully!**

$-5a + 10b = -5(\quad)$

$-6x - 8y = -2(\quad)$

$-15t + 21s = -3(\quad)$

$-24a - 28b = -4(\quad)$

$-2x - x^2 = -x(\quad)$

$-6x + 14z = -2(\quad)$

$-2x + 3y = -1(\quad)$

$-15p + 9r - 12s = -3(\quad)$

$-xy - xz = -x(\quad)$

$-x^2 + x = -x(\quad)$

$22x^2 - 33xy = 11x(\quad)$

$24xy - 16xz = 8x(\quad)$

$35b^2 - 25ab = 5b(\quad)$

$a^4 + a^2 = a^2(\quad)$

$25ab - 75ac = 25a(\quad)$

$x^2y - x^2z = x^2(\quad)$

$2x^2 - 6x = 2x(\quad)$

$6a + 9b - 15c = 3(\quad)$

$a^3 + a^2 - 5a = a(\quad)$

26) **Factorise the following:**

$6x + 9y$

$4a + 10b$

$14p + 21q$

$15a + 25b$

$12k - 15m$

$22c - 33d$

$15x^2 - 25x + 5$

$14y - 7$

$5d + 35e$

$6x^2 + 8xy$

$16xy - 24y^2$

$40a^2 - 5a$

$18t^2 + 27t^3$

$75x^2y + 100xy^2$

$44a^3 - 66a^2$

$24x - 28y + 44z$

$35a - 45b - 15c$

$12a^3b^2 + 15a^2b^3$

$16a^2x^2 - 24ax^3 - 32a^3x$

$72x^4 - 63x^3 - 45x^2 + 81x$

$x^4 - x^2$

$x^3 + x$

$2a + 3a^2$

$x^2 - x$

$4y^6 + 6y^4$

$3x + 6y$

$15a - 10b$

$12m + 16n$

$x^2 - 3x$

$ab - ac$

$2xy - 6xz$

$15b^2 - 25ab$

$x^3 + x^2 - x^4$

$14x^2y - 21xy^2$

$4x^3y^2 - 6x^2y^3$

$ax - ay - cx + cy$

$3x + 3y - ax - ay$

$7x - 7y - kx + ky$

$ak + at - 5k - 5t$

$x^2 - 4x - ax + 4a$

$pq + ps - 9q - 9s$

$ab - ac - b^2 + bc$

$ak - bk + at - bt + a - b$

$3ax - 6ay - x^2 + 2xy$

$15xy - 20y^2 - 9x + 12y$

27) **Re-arrange these and then factorise them:**

$ak + bt + at + bk$

$2ac + bd + 2ad + bc$

$3ax - by + 3bx - ay$

$5ac - 2b - 10a + bc$

$10ac + b - 5bc - 2a$

$ab + 3 + 3a + b$

$15ac + 2bd + 10bc + 3ad$

$3xy - 7 - x + 21y$

$35ab + 6 - 21b - 10a$

$22wx + yz - 2wy - 11xz$

28) **Factorise these ten 'groups-of-four':**

(i) $ma + mb + na + nb$
(ii) $3a + 3b + xa + xb$
(iii) $st + 3t + as + 3a$
(iv) $x^2 - 2x + xy - 2y$
(v) $a^2 + 5a + 3ab + 15b$
(vi) $2x^2 - 5xy + 4kx - 10ky$
(vii) $ac - ad - bc + bd$
(viii) $mx - 2my - 12bx + 24by$
(ix) $pm + nq + np + mq$
(x) $6x^2 + 15ab - 10bx - 9ax$

29) **Factorise the following:**

(i) $xy + 3xz$
(ii) $fx - 3xy - 2mf + 6my$
(iii) $3a^2 - 6a^3$
(iv) $w^2 - 4wy + 12vy - 3vw$
(v) $6a^2 + bc - 2ac - 3ab$
(vi) $14a^2b - 35ab^2$
(vii) $a^3 - ac - a^2k + ck$
(viii) $a^2b^2 - b^3 + bc - a^2c$
(ix) $ax - bx - ay + by + a - b$
(x) $5a^2 + ac - a - 10ab - 2bc + 2b$

30) **Factorise the following:**

1. $x^2 - 9$
2. $x^2 - 16$
3. $x^2 - 25$
4. $a^2 - 100$
5. $a^2 - 49$
6. $y^2 - 64$
7. $p^2 - 9$
8. $m^2 - 81$
9. $k^2 - 36$
10. $9x^2 - 4$
11. $36x^2 - 25$
12. $144x^2 - 49$
13. $4x^2 - 81$
14. $121x^2 - 36$
15. $100a^2 - 9b^2$
16. $a^2 - 25b^2$
17. $4x^2 - 49y^2$
18. $16x^2 - 9y^2$
19. $p^2 - 400q^2$
20. $9m^2 - 25n^2$
21. $a^2b^2 - 1$
22. $m^4 - 9$
23. $x^2y^2 - 36$

31) Factorise these 'difference of two squares':

$c^2 - 25$
$d^2 - 16$
$k^2 - 100$
$4m^2 - 81n^2$
$49k^2 - 9t^2$
$144r^2 - 121s^2$
$36y^2 - z^2$
$x^2 - 1\,000\,000$
$400b^2 - 9c^2$
$a^2b^2 - 4x^2y^2$
$y^6 - 25$
$x^2y^4 - 100$

32) If $x^2 - y^2 = 72$ and $x + y = 9$, write down the value of $(x - y)$.

33) If $a^2 - b^2 = 100$ and $a - b = 5$, write down the value of $(a + b)$.

34) Factorise these 'difference of two squares':

(i) $(x + y)^2 - z^2$
(ii) $(x + 3y)^2 - k^2$
(iii) $(x - y)^2 - w^2$
(iv) $(x + 2y)^2 - 81$
(v) $(a + 2b)^2 - 9c^2$
(vi) $(2x - y)^2 - (x + y)^2$
(vii) $(5a + 4b)^2 - (a + b)^2$
(viii) $4z^2 - (x + y)^2$
(ix) $(3a + 2b)^2 - (2a - 3b)^2$
(x) $(2x - y)^2 - (x + 2y)^2$

35) Find three factors for each of the following, keeping in mind the CF rule (Common Factors Come First).

$2x^2 - 18$
$5x^2 - 20$
$10x^2 - 10$
$7a^2 - 700$
$2y^2 - 32$
$3x^2 - 108$
$x^3 - 4x$
$y^3 - y$
$2x^2 - 8$
$3x^2 - 300$
$5y^2 - 45$
$2k^2 - 98$
$8x^2 - 2$
$50a^2 - 2$
$18a^2 - 8b^2$
$x^3 - 16x$
$x^3 - 4x$
$12ax^2 - 27ay^2$

36) Factorise:
(i) $x^2 - 36$
(ii) $x^2 - x - 72$
(iii) $x^2 - xy + 7x - 7y$
(iv) $px - 2qx - 11p + 22q$
(v) $x^2y^2 - 49$

37) Factorise:
(i) $ax + 6a + px + 6p$
(ii) $x^2 + 3x - 54$
(iii) $y^2 - 9$
(iv) $5y^2 - 45$
(v) $pr + qs + qr + ps$

38) a) Simplify $(a - 4b)(a + b) + 3ab$ and factorise the simplified expression.

b) Simplify $(2t - 9)(t + 8) - 7t$ and factorise fully the simplified expression.

c) Simplify $(a - 9)(ab^2 + 4) - a(2 - 3b)(2 + 3b)$ and factorise the simplified expression.

39) (a) Factorise $4a^2 - 121b^2$
(b) One factor of $x^2 + 8x - 65$ is $(x - 5)$. What is the other?
(c) (i) Copy and fill in the bracket:
$-2a^3b + 4a^2b^2 = -2a^2b(\quad)$
(ii) Factorise
$13a - 26b - 2a^3b + 4a^2b^2$

40) Apply "squared brackets" and factorise the following:

a) $x^2 + 2xy + y^2$
b) $x^2 - 2xy + y^2$
c) $x^2 - 18xy + 81y^2$
d) $x^2 - 6x + 9$
e) $m^2 + 12m + 36$
f) $x^2 + 18x + 81$

41) Factorise the following:

$x^2 + 2xy + y^2 - 81$

$x^2 - 2xy + y^2 - 100$

$x^2 + 4x + 4 - 49y^2$

$a^2 + 6a + 9 - b^2$

$a^2 + 8ab + 16b^2 - c^2$

$a^2 + 2ab + b^2 - 16c^2$

42) a) Factorise $x^2 - 12x + 36$ and hence factorise $x^2 - 12x + 36 - 49y^2$.

b) Factorise $p^2 + 6pq + 9q^2$ and hence factorise:

(i) $p^2 + 6pq + 9q^2 - r^2$

(ii) $p^2 + 6pq + 9q^2 - (r+s)^2$

c) Factorise $x^2 - 13x + 36$ and hence find four factors of $a^4 - 13a^2 + 36$.

d) Factorise $a^2 - 2ab + b^2$ and hence factorise:

(i) $a^2 - 2ab + b^2 - 121c^2$

(ii) $100 - a^2 + 2ab - b^2$

(iii) $(5a - b)^2 - a^2 + 2ab - b^2$

43) Factorise the following and tell the roots of them:

a) $x^4 - x^2$

b) $x^3 + 3x^2 + 4x + 12$

c) $2x^3 - 3x^2$

d) $x^3 - x^2 - 12x$

e) $x^3 - 7x^2 + 14x - 8$

f) $x^4 - 4x^3 + 4x^2 - 4x + 3$

44) Factorise the following using “Ruffini’s rule”:

a) $x^4 + 2x^3 - 23x^2 - 60x$

b) $x^5 + 8x^4 + 21x^3 + 18x^2$

c) $10x^4 - 3x^3 - 41x^2 + 12x + 4$

d) $9x^4 - 36x^3 + 26x^2 + 4x - 3$

e) $x^5 + 10x^4 + 32x^3 + 40x^2 + 31x + 30$

45) **Factorise the following:**

$x^2-7x+10$
$x^2-8x+12$
$x^2-8x+15$
$x^2-9x+20$
x^2-3x+2
$x^2-10x+21$
$x^2-3x-10$
$x^2+2x-15$
x^2-x-12
x^2+x-6
$x^2-4x-12$

$x^2+5x-14$
$x^2-11x+30$
x^2-x-30
x^2+x-30
x^2-4x-5
x^2-4x+4
$x^2+6x-16$
x^2-5x+4
$x^2+9x-22$
$x^2+5x-24$
x^2-x-90

$x^2+8x+12$
$x^2+9x+18$
$x^2+11x+18$
$x^2+10x+25$
x^2+6x+5
$a^2+11a+28$
$y^2+11y+24$
$b^2+14b+33$
x^2+6x+8
$x^2+9x+14$
$x^2+9x+20$

$x^2+7x+12$
x^2+4x+4
x^2+6x+9
x^2+8x+7
$x^2+11x+30$
$x^2+13x+30$
$x^2+17x+30$
$x^2+31x+30$
$p^2+13p+42$
$y^2+14y+49$
$k^2+12k+32$

$x^2+29x+100$
$x^2+14x+48$
$x^2+27x+50$
$x^2+20x+51$
$t^2+13t+36$
$x^2+13x+40$
$x^2+2x-63$
$x^2-15x+56$
x^2+x-42
$x^2+3x-70$
$x^2-10x+24$

46) **Factorise these quadratic expressions:**

$x^2-10x-24$
$x^2+11x-60$
$x^2+2xy-35y^2$
$x^2+xy-6y^2$
$a^2-3ab-40b^2$
$m^2-24mn+144n^2$
$x^2+xy-110y^2$

x^2-x-20
a^2+2a-8
$x^2-7x+12$
y^2+y-42
$b^2+3b-10$
$g^2+4g-12$
$x^2+10x-11$
m^2-6m+5
m^2-5m+6
$k^2-21k-100$

$2x^2-3x+1$
$5x^2+13x-6$
$4x^2+4x+1$
$10a^2+7a+1$
$6y^2-y-1$
$14x^2+3x-2$
$15y^2+13y+2$
$6m^2+m-2$
$18x^2-9x+1$
$18x^2-11x+1$

$x^2+5xy+6y^2$
$a^2+7ab+10b^2$
$3a^2+4ab+b^2$
$10p^2+9pq+2q^2$
$x^2-xy-6y^2$
$2x^2-xz-10z^2$
$7x^2+11xy-6y^2$

47) **Factorise the following:**

(i) $20x^2-x-1$
(ii) $20x^2+x-1$
(iii) $20x^2-8x-1$
(iv) $20x^2+8x-1$
(v) $20x^2-19x-1$
(vi) $20x^2+19x-1$
(vii) $20x^2-12x+1$
(viii) $20x^2-9x+1$
(ix) $20x^2+21x+1$
(x) $1-x-20x^2$

48) **(a)** (i) **Factorise** $x^2+4x-32$
(ii) **Evaluate the expression** $x^2+4x-32$ **when** $x=-8$.
(b) Factorise:
(i) $ab-bc-a+c$
(ii) $x^2+2x-63$

49) (a) Factorise:

(i) $x^2 - 4$

(ii) $x^2 - 4x$

(iii) $x^2 - 4x + 4$

(b) The factors of $x^2 + ax + b$ are $(x-17)$ and $(x+3)$. Find the values of a and b.

(c) Factorise fully:

(i) $7x^2 - 28$

(ii) $x^2 - 49$

(iii) $x^3 - 49x$

50) a) The factors of $x^2 + ax + b$ are $(x-1)$ and $(x+8)$.
Find the values of a and b.

b) The factors of $x^2 + px + q$ are $(x-7)$ and $(x-9)$.
Find the values of p and q.

c) The factors of $x^2 - kx - t$ are $(x-2)$ and $(x+1)$.
Find the values of k and t.

51) Factorise and calculate de LCM (least common multiple) and the GCD (greatest common divisor) of the following:

a) $x^2 - 2x + 1$ and $2x - 2$

b) $x^4 - 4x^2$ and $x^3 - 4x^2 + 4x$

c) $x^2 - 3x$, $x^2 - 9$ and $x^2 - 6x + 9$

Two important theorems about polynomials

Note: the solutions of $a_n x^n + ... + a_2 x^2 + a_1 x + a_0 = 0$ are called the **roots of** $a_n x^n + ... + a_2 x^2 + a_1 x + a_0$ so if the polynomial is factorised $a_n x^n + ... + a_2 x^2 + a_1 x + a_0 = a_n(x-\alpha_1)(x-\alpha_2)...$ the roots are $\alpha_1, \alpha_2, ...$

The remainder theorem

When a polynomial $f(x)$ is divided by $(x - a)$ the remainder is $f(a)$.

Example Find the remainder when the polynomial $f(x) = x^3 + x - 5$ is divided by:

(a) $(x - 3)$ (b) $(x + 2)$ (c) $(2x - 1)$.

(a) The remainder is $f(3) = 3^3 + 3 - 5 = 25$.

(b) The remainder is $f(-2) = (-2)^3 - 2 - 5 = -15$.

(c) The remainder is $f(0.5) = 0.5^3 + 0.5 - 5 = -4.375$.

Choose x so that $x + 2 = 0$.

Choose x so that $2x - 1 = 0$.

The factor theorem

A special case of the remainder theorem occurs when the remainder is 0. This result is known as the **factor theorem** which states:

If $f(x)$ is a polynomial and $f(a) = 0$ then $(x - a)$ is a factor of $f(x)$.

Example Show that $(x + 3)$ is a factor of $x^3 + 5x^2 + 5x - 3$.

Taking $f(x) = x^3 + 5x^2 + 5x - 3$

$$f(-3) = (-3)^3 + 5(-3)^2 + 5(-3) - 3$$
$$= -27 + 45 - 15 - 3 = 0.$$

By the factor theorem $(x + 3)$ is a factor of $x^3 + 5x^2 + 5x - 3$.

Note: the following facts are equivalent:

- α is a solution of $a_n x^n + ... + a_2 x^2 + a_1 x + a_0 = 0$
- the remainder of $(a_n x^n + ... + a_2 x^2 + a_1 x + a_0) : (x-\alpha)$ is zero
- $a_n x^n + ... + a_2 x^2 + a_1 x + a_0$ is divisible by $x-\alpha$
- the polynomial is factorised $a_n x^n + ... + a_2 x^2 + a_1 x + a_0 = (x-\alpha)g(x)$ where $g(x)$ is the quotient

Example A polynomial is given by $f(x) = 2x^3 + 13x^2 + 13x - 10$.

(a) Find the value of $f(2)$ and $f(-2)$.

(b) State one of the factors of $f(x)$.

(c) Factorise $f(x)$ completely.

(a) $f(2) = 2(2^3) + 13(2^2) + 13(2) - 10 = 84$

$f(-2) = 2(-2)^3 + 13(-2)^2 + 13(-2) - 10 = 0.$

(b) Since $f(-2) = 0$, $(x + 2)$ is a factor of $f(x)$ by the factor theorem.

(c) dividing with Ruffini's rule

	2	13	13	−10	
−2		−4	−18	10	divisor (x + 2)
	2	9	−5	0	quotient $2x^2 + 9x - 5$

using now quadratic formula

$$\frac{-9 \pm \sqrt{81+40}}{4} = \begin{cases} -5 \\ 1/2 \end{cases}$$

so we can write

$$f(x) = (x+2)(2x^2 + 9x - 5) = (x+2)2(x+5)\left(x - \frac{1}{2}\right)$$

Factorising a polynomial in order to solve an equation

The equation $a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0 = 0$ can be rewritten as $p(x)q(x)\ldots = 0$ if you factorise the left side of the equation.

This way you can split the equation into several equations easier to solve: $p(x) = 0$, $q(x) = 0$, ...

EXERCISES

52) Tell the roots of $A(x)=(x+5)^2\cdot(2x-3)\cdot x$

53) Tell the roots of $B(x)=(x-2)\cdot(x^2+1)$

54) Find the remainder when x^3+2x^2-5x+1 is divided by $x-2$

55) Let $f(x)=x^5+x^4-19x^3-25x^2+66x+72$
Use the Factor Theorem to determine whether or not $x+1$ is a factor of $f(x)$

56) Find the value of "m" as to $(3x^3+mx^2+x-4):(x-3)$ has a zero remainder

57) Find the value of "m" as to x^3-mx^2+5x-2 be divisible by $x+1$

58) Write a polynomial with the following roots: 2, 3, –1

59) In a division of polynomials the quotient is $2x-3$, the divisor is $3x^2-1$ and the remainder is $-x+1$. Find out the dividend.

60) a) Write a quadratic P(x) such that P(3)=0 and P(5)=6

b) Write a quadratic Q(x) such that Q(–4)=0 and Q(–2)= –8

61) a) Write a quadratic function such that its graph (a parabola) passes through points (3,0) and (5,6)

b) Write a quadratic function such that its graph (a parabola) passes through points (–4,0) and (–2,–8)

62) Find the possible values of "b" if the equation $g(x)=0$ is to have only one root, where $g(x)$ is given by $g(x)=3x^2+bx+12$

63) Solve $x^2-6x+7=0$: a) using the quadratic formula

b) by completing the square

c) by factorising (for example, with Ruffini's rule)

64) a) **Find the values of real numbers p and q if**
$(2x+3)(px+q) = 10x^2+29x+21$
for all values of x.

b) **Find the values of real numbers n and m if**
$(3x-11)(nx+m) = 6x^2-x-77$
for all values of x.

c) **Find the remainder when:**
(i) 660 is divided by 7.
(ii) $(x^2-3x+50)$ is divided by $(x-5)$.
(iii) (x^3-2x^2+6x+1) is divided by $x-1$.

d) **Divide $\dfrac{18x^3+63x^2-8x-28}{2x+7}$ and factorise the result.**

Algebraic fractions (rational expressions)

The quotient of two polynomials is called an **algebraic fraction** (or a **rational expression**). They can be operated like numerical fractions.

They can be simplyfied by cancelling a common factor on the numerator and on the denominator.

To simplify a rational expression, factorise both $f(x)$ and $g(x)$ as far as possible, cancel any factors that appear in both the numerator and denominator.

$$\frac{x^2-9}{x^2-x-6} = \frac{\overset{1}{\cancel{(x-3)}}(x+3)}{\underset{1}{\cancel{(x-3)}}(x+2)} = \frac{x+3}{x+2}$$

Examples:

$$\frac{15a^2}{25a^3}=\frac{3}{5a} \qquad \frac{12xy^3}{18x^4y^2}=\frac{2y}{3x^3} \qquad \frac{x^3}{x^2+x^3}=\frac{x^3}{x^2(1+x)}=\frac{x}{1+x}$$

$$\frac{x^2+x}{yx+y}=\frac{x(x+1)}{y(x+1)}=\frac{x}{y} \qquad \frac{x+1}{x^2+2x+1}=\frac{(x+1)}{(x+1)^2}=\frac{1}{x+1} \qquad \frac{x-1}{x^2-1}=\frac{(x-1)}{(x+1)(x-1)}=\frac{1}{x+1}$$

$$\frac{3x-6}{2x-4}=\frac{3(x-2)}{2(x-2)}=\frac{3}{2} \qquad \frac{x^2-4}{x+2}=\frac{(x+2)(x-2)}{x+2}=x-2 \qquad \frac{x^2-2x+1}{x^2-1}=\frac{(x-1)^2}{(x+1)(x-1)}=\frac{x-1}{x+1}$$

$$\frac{2x^2+6x}{2x^2+7x+3}=\frac{2x(x+3)}{(2x+1)(x+3)}=\frac{2x}{2x+1}$$

This cannot be cancelled any further.

$$\frac{3x^2-12}{2-x}=\frac{3(x^2-4)}{2-x}=\frac{3(x-2)(x+2)}{(2-x)}=-3(x+2)$$

$$\frac{5x+15}{2x^2-18}=\frac{5(x+3)}{2(x^2-9)}=\frac{5\overset{1}{\cancel{(x+3)}}}{2(x-3)\underset{1}{\cancel{(x+3)}}}=\frac{5}{2(x-3)}$$

$$\frac{2a-2b}{b-a}=\frac{2\overset{1}{\cancel{(a-b)}}}{-1\underset{1}{\cancel{(a-b)}}}=\frac{2}{-1}=-2$$

The basic rules are exactly the same as for ordinary fractions and you should definitely be aware of the close similarity.

[redacted]

$\ldots 0w^3 \qquad 6$

$35s^3t^2 \ldots$ / $60w^3 \qquad 12w^2$

$$\frac{12}{p+4} \div \frac{4(p-3)}{3(p+4)} = \frac{\cancel{12}\,3}{\cancel{p+4}} \times \frac{3\cancel{(p+4)}}{\cancel{4}(p-3)} = \frac{9}{p-3}$$

$$\frac{t-2p}{3t-p}-\frac{1}{3}=\frac{3(t-2p)}{3(3t-p)}-\frac{1(3t-p)}{3(3t-p)}$$

$$=\frac{3t-6p-3t+p}{3(3t-p)}=\frac{-5p}{3(3t-p)}$$

Multiplying and dividing algebraic fractions

To multiply several algebraic fractions you must multiply the numerators to get the numerator and multiply the denominators to get the denominator.

To divide two algebraic fractions the rule is to cross-multiply.

Examples: $\frac{2a^2}{5b^3}\cdot\frac{15b}{8a^4}=\frac{2a^2 15b}{5b^3 8a^4}=\frac{3}{4b^2a^2}$ $\qquad 6x^2y\cdot\frac{y}{18x^3}=\frac{6x^2yy}{18x^3}=\frac{y^2}{3x}$

$\frac{3x^2}{5y^4}:\frac{6x^3}{10y^3}=\frac{3x^2 10y^3}{5y^4 6x^3}=\frac{1}{yx}$ $\qquad 1:\frac{x^2}{y^3}=\frac{y^3}{x^2}$

$\frac{x^2-1}{x}:(x-1)=\frac{x^2-1}{x(x-1)}=\frac{(x+1)(x-1)}{x(x-1)}=\frac{x+1}{x}$

$\frac{x^2+2x+1}{x}:\frac{x^2-1}{x^2}=\frac{(x^2+2x+1)x^2}{x(x^2-1)}=\frac{(x+1)^2x^2}{x(x+1)(x-1)}=\frac{(x+1)x}{(x-1)}=\frac{x^2+x}{x-1}$

Getting a common denominator of several algebraic fractions

First calculate the L.C.M. (lower common multiplier) of the denominators.

Change each fraction into another equivalent one whose denominator is the L.C.M. To do so, divide the L.C.M. by the denominator and multiply by the numerator.

Adding or substracting algebraic fractions

Always get a common denominator and then add/substract numerators only.

Examples: $\frac{1}{x}+\frac{2}{x^2}=\frac{x}{x^2}+\frac{2}{x^2}=\frac{x+2}{x^2}$ because L.C.M = x^2

$\frac{3}{x}+\frac{1}{2x}-\frac{5}{3x}=\frac{18}{6x}+\frac{3}{6x}-\frac{10}{6x}=\frac{18+3-10}{6x}=\frac{11}{6x}$ because L.C.M.= 6x

$\frac{5}{2x}-\frac{3}{x^2}=\frac{5x}{2x^2}-\frac{6}{2x^2}=\frac{5x-6}{2x^2}$ because L.C.M.= $2x^2$

$\frac{x}{x+1}-\frac{5}{2(x-1)}=\frac{2(x-1)x}{2(x+1)(x-1)}-\frac{5(x+1)}{2(x+1)(x-1)}=\frac{2x(x-1)-5(x+1)}{2(x+1)(x-1)}=$

because L.C.M.= $2(x+1)(x-1)$

$\frac{2x^2-2x-5x-5}{2(x^2-1)}=\frac{2x^2-7x-5}{2x^2-2}$

$$\frac{x-2}{x^2-x-12}+\frac{4}{x^2-9}=\frac{x-2}{(x+3)(x-4)}+\frac{4}{(x+3)(x-3)}$$

$$=\frac{(x-3)(x-2)+4(x-4)}{(x+3)(x-4)(x-3)}$$

$$=\frac{x^2-5x+6+4x-16}{(x+3)(x-4)(x-3)}$$

$$=\frac{x^2-x-10}{(x+3)(x-4)(x-3)}$$

$$\frac{3x-1}{2}+\frac{5x+2}{3}-\frac{11x-1}{4}=\frac{6(3x-1)+4(5x+2)-3(11x-1)}{12}$$
$$=\frac{18x-6+20x+8-33x+3}{12}$$
$$=\frac{5x+5}{12}$$

$$\frac{3}{2x+5}-\frac{5}{7x-2}=\frac{3(7x-2)-5(2x+5)}{(2x+5)(7x-2)}$$
$$=\frac{21x-6-10x-25}{(2x+5)(7x-2)}$$ (Watch those signs!)
$$=\frac{11x-31}{(2x+5)(7x-2)}$$

"Check" of last example

Put $x = 2$ at the start and the end:

Start: $\frac{3}{2x+5}-\frac{5}{7x-2}=\frac{3}{9}-\frac{5}{12}=\frac{1}{3}-\frac{5}{12}=\frac{4-5}{12}=-\frac{1}{12}$

End: $\frac{11x-31}{(2x+5)(7x-2)}=\frac{22-31}{(9)(12)}=-\frac{9}{108}=-\frac{1}{12}$

Since these are the same, the answer is reinforced.

EXERCISES

65) Reduce these fractions by *dividing* above and below by a common factor:

1. $\frac{4}{10}$
2. $\frac{3a}{6a}$
3. $\frac{2a+2b}{3a+3b}$
4. $\frac{2x-4}{x^2-4}$
5. $\frac{a-3b}{3a-9b}$
6. $\frac{8x-10}{16x^2-25}$
7. $\frac{x^2-9}{x^2-4x+3}$
8. $\frac{x-4}{x^2-16}$
9. $\frac{x^2-100}{5x+50}$
10. $\frac{a-b}{b-a}$
11. $\frac{a^2-b^2}{7a-7b}$
12. $\frac{x+4}{16-x^2}$
13. $\frac{a-6}{36-a^2}$
14. $\frac{x^2-2x}{x^2-4}$
15. $\frac{6a-6b}{a^2-b^2}$
16. $\frac{2x-2y}{3y-3x}$
17. $\frac{2x^2+9x+4}{2x^2+11x+5}$
18. $\frac{by+b-y-1}{b^2-1}$
19. $\frac{ax+ay-cx-cy}{ax+ay+cx+cy}$
20. $\frac{9y^3-y}{3y^2+8y-3}$
21. $\frac{x^2+x-6}{x^2+3x-10}$
22. $\frac{a^2-b^2}{a^2-2ab-3b^2}$
23. $\frac{ab-ay-bc+cy}{b^2-y^2}$
24. $\frac{a-b}{b^2-a^2}$
25. $\frac{x^2-2xy+y^2}{x^2-y^2}$

66) Simplify:

i) $\frac{2x^2+3x}{2x^2+x-3}$ ii) $\frac{9x-x^3}{3x^2+x^3}$ iii) $\frac{a^3+7a^2+14a+8}{(a+1)(a+2)}$ iv) $\frac{(2a-1)(a^2+a-3)+4a(2a-1)}{(2a-1)}$

v) $\frac{(y^2-1)^2+5y(y^2+2y+1)}{(y+1)^2}$

67) Simplify $\frac{5x(8x^2-2)-6(1-4x^2)}{(5x+3)}$

and factorise fully the simplified expression.

68) Write these as a single fraction:

1. $\frac{x+1}{3}+\frac{x+5}{2}$

2. $\frac{x+3}{5}+\frac{x+1}{2}$

3. $\frac{x+1}{2}+\frac{x+2}{3}+\frac{x+3}{4}$

4. $\frac{x+2}{6}+\frac{x+1}{3}+\frac{x+5}{2}$

5. $\frac{x+5}{3}+x+4$ (Hint: $x+4 = \frac{x+4}{1}$)

6. $x+\frac{2x+3}{7}$

7. $3x+1+\frac{5x-1}{3}$

8. $\frac{5x-1}{2}-\frac{x-1}{3}$

9. $\frac{7x-2}{10}-\frac{2x-1}{5}+\frac{x+3}{2}$

10. $\frac{5x-3}{4}+\frac{2x-1}{2}-\frac{4-x}{8}$

11. $\frac{1}{x+1}+\frac{1}{x+4}$

12. $\frac{3}{x+1}+\frac{2}{x+2}$

13. $\frac{2}{2x+1}+\frac{5}{3x+5}$

14. $\frac{3}{x-3}+\frac{5}{2x+4}$

15. $\frac{7}{2y+1}-\frac{6}{2y-1}$

16. $\frac{7}{2a-3}-\frac{4}{3a-2}$

17. $\frac{11}{7y-10}-\frac{2}{3y-1}$

18. $\frac{3}{6n+1}-\frac{1}{2n-3}$

19. $\frac{3}{6y-1}-\frac{2}{4y+7}$

20. $\frac{1}{1-x}+\frac{3}{x-3}$

69) a) Write $\frac{3}{x-1}+\frac{4}{x+4}$ as a single fraction and verify your answer by putting $x = 4$.

b) Write $\frac{15}{x+2}-\frac{12}{x+5}$ as a single fraction. Verify your answer by putting $x = 1$.

c) Write $\frac{8}{x-5}-\frac{12}{5-x}$ as a single fraction. Verify your answer by letting $x = 9$.

d) Write $\frac{7}{x-3}+\frac{2}{3-x}$ in the form $\frac{k}{x-3}$, where $k \in R$.
(Hint: The lowest common denominator is just $(x-3)$.)

e) Write $\frac{5}{2x-1} \quad \frac{4}{3x+1}$ as a single fraction.
Verify your answer by letting $x = 3$

70) Express as a single fraction in its simplest form:

(a) $\frac{3(x-4)}{(x+2)(x-1)} + \frac{2(x+1)}{x-1}$

(b) $\frac{x^2-3x+2}{x^2+3x-4} - \frac{1}{(x+4)^2}$

(c) $\left[\left(x+\frac{1}{x}\right) : \left(x-\frac{1}{x}\right)\right] \cdot (x-1)$

(d) $\frac{2}{x} \cdot \left(\frac{1}{x} : \frac{1}{x-1}\right)$

(e) $\left(\frac{x-1}{x^2} + \frac{3}{x} - \frac{5}{x-4}\right) \cdot 2x^2$

(f) $\left(\frac{3}{x} - \frac{x}{3}\right) : \left(\frac{1}{x} + \frac{1}{3}\right)$

(g) $\frac{x+1}{(x-1)^2} \cdot \frac{x^2-1}{x}$

71) (i) Evaluate $\frac{3}{x+1} + \frac{10}{x+2}$ when $x = \frac{1}{2}$.

(ii) Write $\frac{3}{x+1} + \frac{10}{x+2}$ as a single fraction.

(iii) Find the value of this single fraction when $x = \frac{1}{2}$.

72) Show that $\frac{7}{2x-4} + \frac{5}{3x-6}$ can be written in the form $\frac{k}{6(x-2)}$ and find the value of k.

73) (a) Find a common factor in the two expressions $(6x-12)$ and $(5x-10)$.

(b) Hence solve

$$\frac{x+2}{5x-10} - \frac{4x-3}{6x-12} + \frac{x+3}{12} = 0$$

74) (a) Factorise $2x^2 - x - 10$

(b) Solve

$$\frac{7}{2x-5} + \frac{4}{x+2} = \frac{x^2+20}{2x^2-x-10}$$

75) Solve for x: $\frac{x+2}{x-4} - \frac{x+3}{x-2} = 1$

$$\frac{16}{x-1} - \frac{14}{x+1} = 3 + \frac{12}{x^2-1}$$

$$\frac{5}{x-4} - \frac{1}{2x+1} = \frac{(x-1)(x+1)}{2x^2-7x-4}$$

$$\frac{3}{x+1} + \frac{4}{x-1} + \frac{2}{x+2} = 0$$

$$\frac{x-2}{x^2-3x} - \frac{x-1}{x^2-2x} = \frac{1}{12x}$$

76) Find the two values of x which satisfy: $\frac{x+1}{x-2} = \frac{1}{2} + \frac{x+2}{x-1}$ and verify that both solutions are correct.

77) Solve $\frac{12}{x+2} - \frac{2}{x+1} = \frac{x^2+8x+9}{x^2+3x+2}$ and verify your solution.

a) Find, correct to one decimal place, the solution of $\frac{x}{x-1} + \frac{x-1}{x} = 4$.

b) Find, correct to two decimal places the solution to the equation

$$\frac{2x-3}{2x-2} - \frac{2x-2}{2x-3} = 1.$$

c) Solve for x, correct to three decimal places, $\frac{2(x-2)}{x-3} + \frac{3x-2}{x-2} = 3$.

d) Find, correct to two decimal places, the values of x which satisfy the equation

$$\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{2x+1} = 0.$$